Working in $\mathbb{R}^3(\mathbb{R})$

Given $L = \{\bar{x} \in \mathbb{R}^3 / x^1 + x^2 + x^3 = 0\}$ prove L is a subspace of $\mathbb{R}^3$ (giving also base and dim.)

L is a subspace of $\mathbb{R}^3 \iff \alpha\bar{x} + \beta\bar{y} \in L \quad \forall \alpha, \beta \in \mathbb{R} \wedge \bar{x}, \bar{y} \in L$

if $\bar{x} \in L \rightarrow x^1 + x^2 + x^3 = 0$ ①

if $\bar{y} \in L \rightarrow y^1 + y^2 + y^3 = 0$ ②

so if $\underbrace{\alpha\bar{x} + \beta\bar{y}}$ $\in L$ then $\alpha x^1 + \beta y^1 + \alpha x^2 + \beta y^2 + \alpha x^3 + \beta y^3 =$

$\alpha\bar{x} + \beta\bar{y} = (\alpha x^1 + \beta y^1, \alpha x^2 + \beta y^2, \alpha x^3 + \beta y^3)$

$= \alpha(\underbrace{x^1 + x^2 + x^3}_{0 \text{ because ①}}) + \beta(\underbrace{y^1 + y^2 + y^3}_{0 \text{ because ②}}) =$

$= \alpha \cdot 0 + \beta \cdot 0 = 0 \rightarrow$ So L is a subspace

$$x^1 + x^2 + x^3 = 0 \longrightarrow \begin{cases} x^1 = \alpha \\ x^2 = \beta \\ x^3 = -\alpha - \beta \end{cases} \quad \forall \alpha, \beta \in \mathbb{R}$$

$\dim(L) = 2$

$$B_L = \{\overbrace{(1,0,-1)}^{\bar{w}_1}, \overbrace{(0,1,-1)}^{\bar{w}_2}\}$$

$$L = \mathcal{L}\{\underbrace{(1,0,-1)}_{\bar{w}_1}, \underbrace{(0,1,-1)}_{\bar{w}_2}\}$$

Obtain the Gramm Matrix of the scalar product $\bar{x}\cdot\bar{y} = x^1y^1 + 2x^2y^2 + 3x^3y^3$ in base B. (proving in first hand that it is a scalar product). $B = [\bar{e}_1, \bar{e}_2, \bar{e}_3]$

Symmet. $\bar{x}\cdot\bar{y} = x^1y^1 + 2x^2y^2 + 3x^3y^3 = y^1x^1 + 2y^2x^2 + 3y^3x^3 = \bar{y}\cdot\bar{x}$ ✓

Lineal. $(\alpha\bar{x}+\beta\bar{y})\cdot\bar{z} = (\alpha x^1+\beta y^1)z^1 + 2(\alpha x^2+\beta y^2)z^2 + 3(\alpha x^3+\beta y^3)z^3 = \alpha(x^1z^1 + 2x^2z^2 + 3x^3z^3) +$

$+\beta(y^1z^1 + 2y^2z^2 + 3y^3z^3) = \alpha(\bar{x}\cdot\bar{z}) + \beta(\bar{y}\cdot\bar{z})$ ✓

Positiv. $\bar{x}\cdot\bar{x} = \underbrace{(x^1)^2}_{>0} + 2\underbrace{(x^2)^2}_{>0} + 3\underbrace{(x^3)^2}_{>0} > 0$ ✓

$$G_B = \begin{pmatrix} \bar{e}_1\cdot\bar{e}_1 & \bar{e}_1\cdot\bar{e}_2 & \bar{e}_1\cdot\bar{e}_3 \\ \bar{e}_1\cdot\bar{e}_2 & \bar{e}_2\cdot\bar{e}_2 & \bar{e}_2\cdot\bar{e}_3 \\ \bar{e}_1\cdot\bar{e}_3 & \bar{e}_2\cdot\bar{e}_3 & \bar{e}_3\cdot\bar{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Obtain $\omega(L)$ orthogonal subspace to L using the previously defined subspace L and scalar product (G_B).

$$\omega(L) = \left\{ \bar{x} \in \mathbb{R}^3 \,/\, \bar{x}\cdot\bar{y} = 0 \quad \forall \bar{y} \in L \right\} = \left\{ \bar{x} \in \mathbb{R}^3 \,/\, \bar{x}\cdot\bar{w}_1 = 0 \wedge \bar{x}\cdot\bar{w}_2 = 0 \;;\; B_L = \{\bar{w}_1, \bar{w}_2\} \right\}$$

$$B_L = \{ \underbrace{(1,0,-1)_B}_{\bar{w}_1}, \underbrace{(0,1,-1)_B}_{\bar{w}_2} \}$$

$$G_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\bar{x}\cdot\bar{w}_1 = 0 \longrightarrow (x^1\ x^2\ x^3)_B\, G_B \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_B = 0 \rightarrow x^1 - 3x^3 = 0$$

$$\bar{x}\cdot\bar{w}_2 = 0 \longrightarrow (x^1\ x^2\ x^3)_B\, G_B \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}_B = 0 \longrightarrow 2x^2 - 3x^3 = 0$$

$$\dim(\omega(L)) = 1$$

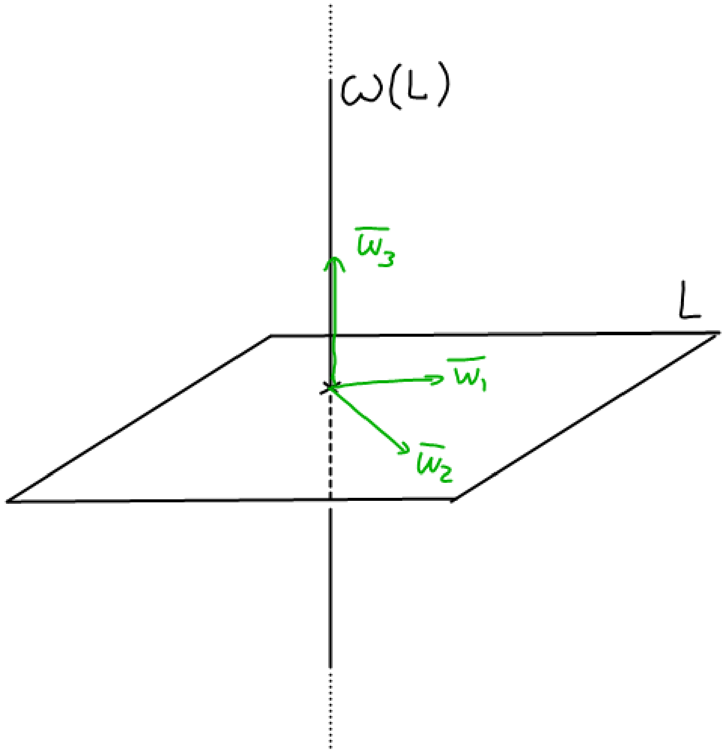
$$\omega(L) \equiv \begin{cases} x^1 - 3x^3 = 0 \\ 2x^2 - 3x^3 = 0 \end{cases} \longrightarrow \begin{cases} x^1 = 3\gamma \\ x^2 = \frac{3}{2}\gamma \quad \forall \gamma \in \mathbb{R} \\ x^3 = \gamma \end{cases}$$

$$B_{\omega(L)} = \{ \underbrace{(6,3,2)_B}_{\bar{w}_3} \}$$

L and $\omega(L)$

$\boxed{\omega(L) \oplus L}$ ALWAYS

$$\omega(L) \oplus L \quad \begin{cases} L + \omega(L) = \mathbb{R}^3 \\ L \cap \omega(L) = \{\bar{0}\} \end{cases}$$



$$B_L = \{ \underbrace{(1,0,-1)_B}_{\overline{w}_1}, \underbrace{(0,1,-1)_B}_{\overline{w}_2} \}$$

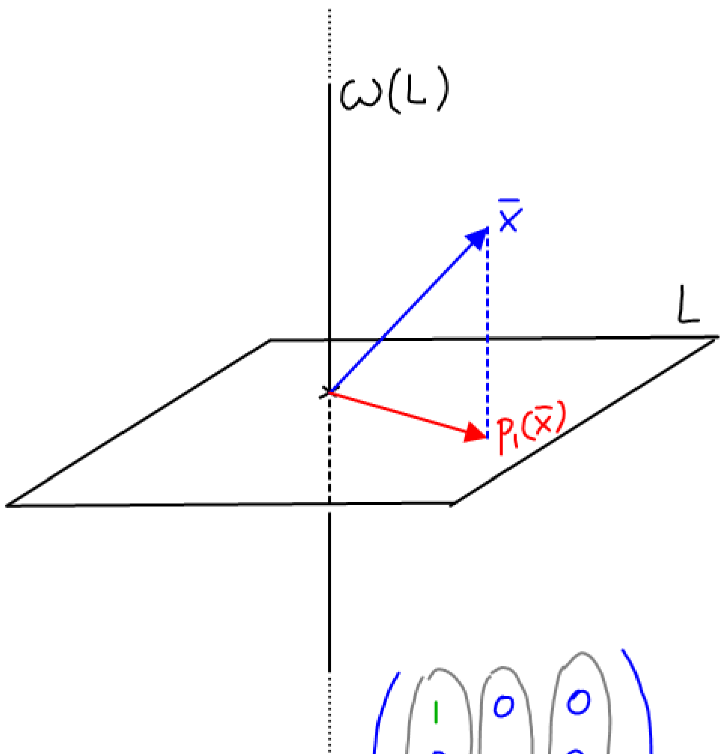
$$B_{\omega(L)} = \{ \underbrace{(6,3,2)_B}_{\overline{w}_3} \}$$

$$B' = B_L \cup B_{\omega(L)} = \{ \underbrace{\overline{w}_1, \overline{w}_2}_{\in L}, \underbrace{\overline{w}_3}_{\in \omega(L)} \} \equiv \text{Projection Base}$$

$$\begin{tikzcd} B \arrow[r, bend left, "C"] & B' \arrow[l, bend left, "C^{-1}"] \end{tikzcd}$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ -1 & -1 & 2 \end{pmatrix} \\ \begin{array}{ccc} \overline{w}_1 & \overline{w}_2 & \overline{w}_3 \end{array} \\ \text{in } B \end{array} \xrightarrow{C^{-1} = \frac{Adj(C)^t}{|C|}} C^{-1} = \begin{array}{c} \begin{pmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{pmatrix} \\ \begin{array}{ccc} \overline{e}_1 & \overline{e}_2 & \overline{e}_3 \end{array} \\ \text{in } B' \end{array} = \frac{1}{11} \begin{pmatrix} 5 & -6 & -6 \\ -3 & 8 & -3 \\ 1 & 1 & 1 \end{pmatrix}$$

Projection of a vector $\bar{x}$ on L



Obtain $p_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ projection endomorph. on L. Calculate its Eigenvalues and Eigenspaces.

Project vector $\bar{a} = (1,1,1)_B$ on L.

$$(P_1)_B = \begin{array}{c} \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix} \\ \begin{matrix} p_1(\bar{e}_1) & p_2(\bar{e}_2) & p_2(\bar{e}_3) \end{matrix} \\ \text{in } B \end{array}$$

$$(P_1)_B \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_B}_{\bar{a}} = \frac{1}{11} \underbrace{\begin{pmatrix} -7 \\ 2 \\ 5 \end{pmatrix}_B}_{p_1(\bar{a})}$$

$$(P_1)_{B'} = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} p_1(\bar{w}_1) & p_1(\bar{w}_2) & p_1(\bar{w}_3) \end{matrix} \\ \text{in } B' \end{array}$$

$p_1(\bar{w}_1) = \bar{w}_1 = (1,0,0)_{B'}$

$p_1(\bar{w}_2) = \bar{w}_2 = (0,1,0)_{B'}$

$p_1(\bar{w}_3) = \bar{0} = (0,0,0)_{B'}$

$(P_1)_{B'} = C^{-1} (P_1)_B \, C$

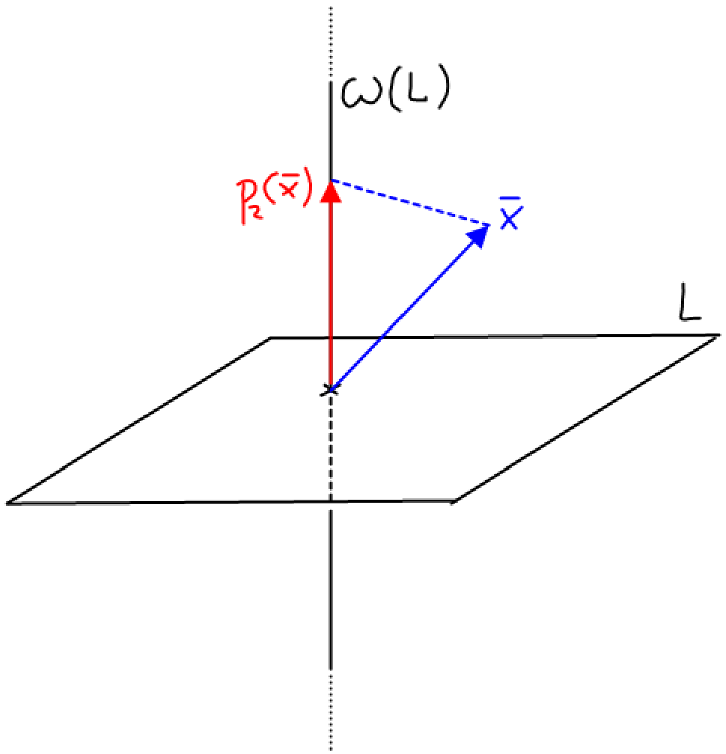
$(P_1)_B = C \, (P_1)_{B'} \, C^{-1}$

$S(1) = L$

$S(0) = \omega(L)$

$$(P_1)_B = \frac{1}{11} \begin{pmatrix} 5 & -6 & -6 \\ -3 & 8 & -3 \\ -2 & -2 & 9 \end{pmatrix}$$

Projection of a vector $\bar{x}$ on $\omega(L)$



Obtain $p_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ projection endomorph. on $\omega(L)$ in the projection base B'.

Calculate its Eigenvalues and Eigenspaces.

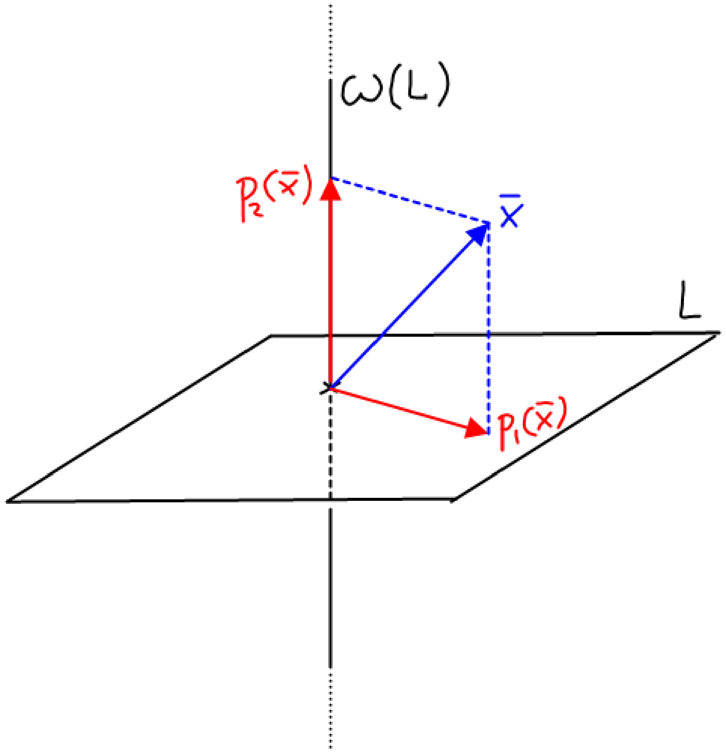
$$
(P_2)_{B'} = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} p_2(\bar{w}_1) & p_2(\bar{w}_2) & p_2(\bar{w}_3) \end{array} \end{array}
$$

in B'

$S(0) = L$

$S(1) = \omega(L)$

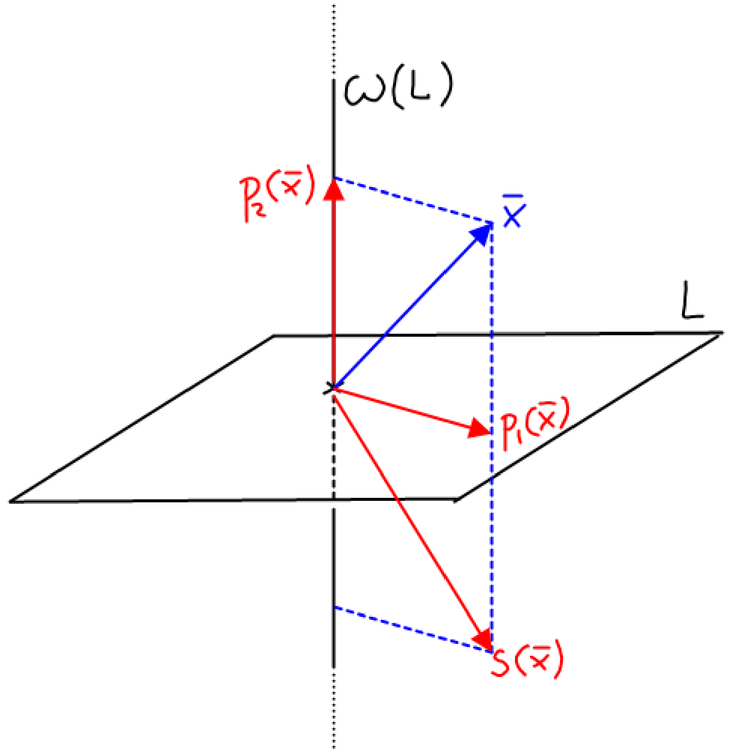
Projection of a vector $\bar{x}$ on L and $\omega(L)$



Since $\boxed{\omega(L) \oplus L}$ ALWAYS

$\forall\, \bar{x} \in \mathbb{E} \qquad \bar{x} = P_1(\bar{x}) + P_2(\bar{x})$

Projection of a vector $\bar{x}$ on L and $\omega(L)$



Obtain $S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ symmetry endomorph. against L in the projection base B'.

Calculate it's Eigenvalues and Eigenspaces.

$$S_{B'} = \begin{array}{ccc} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right) \\ S(\bar{w}_1) \; S(\bar{w}_2) \; S(\bar{w}_3) \end{array}$$

in B'

$S(1) = L$

$S(-1) = \omega(L)$